INVARIANT SUBSPACES OF A MEASURE PRESERVING TRANSFORMATION*

BY

S. R. FOGUEL

ABSTRACT

We consider, in this note, some invariant subspaces of a unitary operator induced by a measure preserving transformation. For these subspaces two problems are studied:

a. Is the subspace generated by characteristic functions?

b. When is an invariant subspace a reducing subspace?

1. **Introduction.** Let (Ω, Σ, μ) be a measure space with $\mu(\Omega) = 1$. Let ϕ be a one-to-one measure preserving transformation. The transformation ϕ induces a unitary operator U, on $L_2(\Omega, \Sigma, \mu)$. We shall consider the following subspaces of $L_2(\Omega, \Sigma, \mu)$:

$H_0 = \{x \mid x \in L_2, \text{weak} \lim_{n\to\infty} U^n x = 0\}$

$H_1 = H_0^{\perp}$

$H_2 = \text{span } H'_2$, where $H'_2 = \{x \mid x \in L_2, \limsup |(U^n x, x)| = \| x \|^2\}$

$H_3 = \{x \mid x \in L_2, \textit{the orbit } U^n x, n = 1, 2, \cdots, \textit{is conditionally compact}\}$.

Let us summarize some properties of these spaces:

1. The subspaces reduce the operators U. Theorem 3.1 of [2].

2. $H_3 \subset H_2 \subset H_1$. See Theorem 1.3 of [2].

3. The subspace H_3 is generated by eigenfunctions of U. This is a well-known result (see [6] page 24) let us sketch its proof: Clearly every eigenfunction belongs to H_3. Let x be orthogonal to all eigenfunctions of U.

By [5] page 40 there exist dense (complements of sets of zero density) sequences $n_i^{(k)}$ with $\lim (U^{n_i(k)} x, U^k x) = 0$. Since the intersection of two dense sequences is dense it is possible to emplyo the diagonal method to find a sequence n_i with $\lim (U^{n_i} x, U^k x) = 0$, $k = 1, 2, \cdots$. Thus weak $\lim U^{n_i} x = 0$.

Let y be any element of H_3. We may assume that $U^{n_i} y$ (or a subsequence) converges strongly to z. Then:

$$\| U^{*n_i} z - y \|^2 = 2\| y \|^2 - 2 \operatorname{Re}(z, u^{n_i} y) \to 0.$$

Therefore

$$(x, y) = \lim (x, U^{*n_i} z) = \lim (U^{n_i} x, z) = 0.$$

---

* This work was partially supported by N.S.F. Grant No. GP 2491.

Received October 2, 1964.

4. The operator U is strongly mixing if and only if $H_1 =$ constant functions, and is weakly mixing if and only if $H_3 =$ constant functions. See [5] for definitions of mixing properties.

2. **The space H_2.** Let $x \in H_2'$, then there exists a subsequence, n_i of the integers, such that $\lim U^n x = x$. (Theorem 1.3 of [2]) Define

$$K = \{y \mid \lim U^{n_i} y = y\} = \{y \mid \lim (U^{n_i} y, y) = \| y \|^2\}.$$

Thus $K \subset H_2'$. If y is a function in K then so are the real and imaginary parts of y. Also if y is a real valued function in K then so is its absolute value:

$$\lim (U^n \mid y \mid, \mid y \mid) \geqq \lim \mid (U^{n_i} y, y) \mid = \| y \|^2 = \| \mid y \mid \|^2.$$

Thus K satisfies the conditions of Lemma 1 of [3]. Therefore K is spanned by the characteristic functions that belong to it. This also proved:

THEOREM 2.1. *The set H_2' is generated by the characteristic functions that belong to it.*

Given a set $A \in \Sigma$ let $I(A)$ be its characteristic function. Then $I(A) \in H_2'$ if and only if

$$\limsup \mu(\varphi^{-n} A \cap A) = \mu(A).$$

D. Austin suggested the following conjecture:

If $\limsup \mu(\varphi^{-n} A \cap A) < \mu(A)$ *whenever* $0 < \mu(A) < 1$ *then* φ *is strongly mixing.*

By Theorem 2.1 this can be rephrased to:

If H_2 contains only constant functions so does H_1.

Conjecture. $H_1 = H_2$ *for every measure preserving transformation.*

It was proved by S. Kakutani that Austin's condition implies that φ is weakly mixing. This can be deduced from Theorem 2.1. since if $Ux = \lambda x$ with $\mid \lambda \mid = 1$ then $\mid (U^n x, x) \mid = \| x \|^2$ and $x \in H_2'$.

3. **Invariant subspaces of H_1.** In this section it is not necessary to assume that U is induced by a measure preserving transformation. The spaces H_1, H_2 and H_3 are well defined for any unitary operator U on a Hilbert space H.

THEOREM 3.1. *Let $M = M(x)$ be the subspace generated by weak limits of the sequence $U^n x$. Then $x \in H_1$ if and only if $x \in M$.*

Proof. Let $z \in H_0(U)$ then $\lim (U^n z, z) = 0$. Conversely if $(U^n z, z) = 0$ and weak $\lim U^{n_i} z = z_1$ then $(z_1, U^k z) = \lim (U^{n_i - k} z, z) = 0$ hence $z_1 = 0$ and weak $\lim U^n z = 0$. Thus $H_0(U) = H_0(U^*)$. Therefore if weak $\lim U^{n_i} x = y$ then $y \perp H_0(U^*) = H_0(U)$ and $y \in H_1$. This shows that if $x \in M$ then $x \in H_1$. Conversely let $x \in H_1$. Put $x = x_1 + x_2$ where $x_1 \in M$ and $x_2 \perp M$. Now $x \in H_1$ and $x_1 \in M \subset H$ hence $x_2 \in H_1$ too. Also $\lim\limits_{n \to \infty} (U^n x, x_2) = 0$ because if $U^{n_i} x$ converges weakly to y then $y \in M$. We wish to show that this implies that $(x, x_2) = \| x_2 \|^2 = 0$. Let $x = y_1 + y_2$ where $(y_1, U^n x_2) = 0$ $n = 0, +1, +2, \cdots$ and y_2 is in the subspace generated by $U^n x_2$ $n = 0, \pm 1, \pm 2, \cdots$. It is enough to show that $y_2 = 0$:

By assumption $\lim (U^n y_2, x_2) = 0$ hence

$$\lim_{n\to\infty} (U^n y_2, u^k x_2) = \lim_{n\to\infty} (U^{n-k} y_2, x_2) = 0 \quad k=0, \pm 1, \pm 2, \cdots.$$

By taking linear combinations and passing to limit one gets $\lim (U^n y_2, y_2) = 0$ or $y_2 \in H_0(U)$. On the other hand $y_2 \in H_1(U)$ since $x_2 \in H_1$ and $U^n x_2 \in H_1$ for every n, thus $y_2 = 0$.

REMARK. We have reproduced in this proof, for the sake of completeness, parts of Theorem 1 of [4] and Theorem 3.1 of [2].

COROLLARY. *Every invariant subspace of H_1 reduces U.*

In Corollary of Proposition 3 of [1] this result is proved for subspaces of H_3.

If $x \in H_1$ then $U^k x \in M$ for any integer k, since M is invariant under U^k.

A similar characterization exists for H_2'.

THEOREM 3.3. *Let K be the close convex hull of the weak limits of the sequence $U^n x$. Then $x \in H_2'$ if and only if $x \in K$.*

Proof. If $x \in H_2'$ then it is the strong limit of $U^{n'} x$, for some subsequence of the integers, by Theorem 1.3 of [2]. Conversely if $x \in 'H_2'$ let

$$\limsup |(U^n x, x)| = \alpha < \| x \|^2.$$

If $y_1 \cdots y_k$ are in K and a_i are positive numbers whose sum is 1 then

$$|(\Sigma a_i y_i, x)| \leqq \Sigma a_i |(y_i, x)| \leqq \alpha < \| x \|^2$$

and no such sum can approximate x.

REFERENCE

1. R. L. Adler, *Invariant and reducing subalgebras of measure preserving transformations*, Trans. Amer. Math. Soc. **110** (1964), 350–361.

2. R. S. Fogel, *Powers of a contraction in Hilbert space*, Pacific J. Math. **13** (1963), 551–562.

3. ———, *On order perserving contractions*, Israel J. Math. **1** (1963), 54–59.

4. ———, *Weak limits of powers of a contraction in Hilbert space*, Proc. Amer. Math. Soc. (to appear).

5. R. P. Halmos, *Lectures on ergodic theory*, Math. Soc. Japan, No. 3 (1956).

6. K. Jacobs, *Ergodentheorie*, Springer, Berlin 1960.

NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS
AND
THE HEBREW UNIVERSITY OF JERUSALEM